The Influence of Bed Friction on Long Wave Run-up

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Abstract: The influence of bed friction on long wave run-up is investigated in this study using a lattice Boltzmann model for the one-dimensional (1D) shallow water equations. In order to appropriately simulate the wetting and drying boundary, the lattice Boltzmann approach which contains bed friction directly is adopted. A 1D tidal wave over three different bed slopes is used to test the water surface changes in varying bed friction coefficient. The result implies that the bigger bed friction, the lower wave run-up or run-down speed.

Keywords: lattice Boltzmann model; bed friction; long wave run-up; wetting and drying boundary.

1. Introduction

Shallow water flows exist widely in nature, such as rivers, coastal areas, estuaries, etc. The shallow water flow usually involves a common and complex phenomenon that is long wave run-up, which is vitally important in coastal and ocean engineering. For instance, when the huge of this type waves climb to near-shore land, which will often induce significant losses of lives and property at the coastal regions. Thus, the study of long wave run-up in shallow waters has aroused widespread concern in academic circles. Many scholars have carried out research in this area.

As early as 1953, Hall and Watts [1] did experiments about solitary wave run-up on an impermeable slope and established an empirical formula. Saville [2] conducted a physical model test for regular waves climbing on the slope. Carrier and Greenspan [3] obtained the analytical solutions for non-breaking waves run-up through the shallow water equations into second order linear equations. Synolakis [4] gained a simple power law through investigating the run-up of long waves in theory. These previous methods are either on the base of flume experiments or theoretical research, and their applications are limited. Recent years, with the development of the computer, numerical methods have been put forward.

Kobayashi et al. [5] employing an explicit dissipative Lax-Wendroff finite-difference method, created a numerical flow model to predict the flow characteristics on rough slopes. Leclerc et al. [6] proposed a two-dimensional (2D) finite element model to study the tidal wave run-up. Hu et al. [7] presented a 1D high-resolution finite volume model to simulate wave propagation. Hubbard and Dodd [8] improved a 2D numerical model of wave run-up using an upwind finite volume technique and a hierarchical cartesian adaptive mesh refinement algorithm. Jamois et al. [9] utilized a finite difference model based on a highly-accurate Boussinesq-type formulation to study wave run-up. In addition to these traditional numerical methods to simulate long wave run-up, the lattice Boltzmann method (LBM) on the strength of easy programming, inherent parallel features and effective treatment for complex boundary conditions, has attracted many researchers [10, 11, 12].

Another important phenomenon is that the wetting and drying boundary, caused by the long wave run-up, plays a vital role in practical projects. Zelt [13] utilized a Lagrangian form of the Boussinesq-type equation to simulate moving boundary. Madsen et al. [14] employed the permeable slot method [15] with a Boussinesq-type model. Kennedy et al. [16] improved the “slot method”, which reduced the loss of water for populating the slit. Lynett et al. [17] presented a linear extrapolation scheme, which is a pure mathematical method. Frandsen [18] incorporated the thin film and the liner extrapolation scheme to treat the wet-dry interface. Liu and Zhou [19] proposed a lattice Boltzmann approach to simulate the wetting and drying front.

However, most of the numerical methods are not considered the effect of external force terms. Clearly, external force such as bed friction is very important in the simulations, which will impact the boundary movement that directly affect the accuracy of the solution. Therefore, the objective of the present study is to investigate the influence of bed friction on long wave run-up. The lattice Boltzmann model for 1D shallow water...
equations using the lattice Boltzmann approach to treat the wet-dry boundary is adopted in this paper. A 1D tidal wave over three adverse bed slopes is applied to simulate the influence of bed friction.

2. Method

2.1. Lattice Boltzmann model for shallow water equations

In this part, the 1D shallow water equations are introduced. Then the lattice Boltzmann equation is described to solve the shallow water equations based on D1Q3 (one-dimensional three-velocity) lattices.

The 1D shallow water equations with bed friction can be written as

\[ \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \]  
\[ \frac{\partial (hu)}{\partial t} + \frac{\partial (hu^{2})}{\partial x} = -\frac{g}{2} \frac{\partial h^{2}}{\partial x} + \nu \frac{\partial^{2} (hu)}{\partial x^{2}} + F, \]

where \( \tau \) and \( u \) are the distance and the velocity, respectively; \( h \) is water depth; \( g \) is the gravitational acceleration; \( \nu \) is the kinematic viscosity; and \( F \) is the force defined as

\[ F = -g \rho \frac{\partial z_{b}}{\partial x} - \frac{\tau_{b}}{\rho}, \]

in which \( z_{b} \) is bed elevation above datum; \( \rho \) is fluid density; the bed shear stress \( \tau_{b} \) is determined by the depth-averaged velocity

\[ \tau_{b} = \rho C_{b} |u|, \]

and \( C_{b} \) is the bed friction coefficient may be either constant or derived from \( C_{b} = g n_{k}^{2} h^{5} / n_{k}^{2} \); \( n_{k} \) is the Manning’s coefficient.

The lattice Boltzmann equation on D1Q3 can be expressed as [20]

\[ f_{\alpha}(x + e_{\alpha} \Delta x, t + \Delta t) = f_{\alpha}(x, t) - \frac{1}{\tau} [f_{\alpha}(x, t) - f_{\alpha}^{eq}(x, t)] + \frac{\Delta t}{N_{\alpha} \epsilon_{\alpha}^{2}} \epsilon_{\alpha} F, \]

in the above equation, \( f_{\alpha} \) is the particle distribution function; \( \tau \) is the Bhatnagar-Cross-Krook (BGK) single relaxation term; \( e = \Delta x / \Delta t; \Delta x \) is the lattice size; \( \Delta t \) is the time step; \( \epsilon_{\alpha} \) is the velocity vector of a particle in link \( \alpha \), i.e. \( \epsilon_{1} = e, \epsilon_{2} = -e, \epsilon_{3} = 0; \) \( N_{\alpha} \) is a constant decided by the lattice pattern as

\[ N_{\alpha} = \frac{1}{\epsilon_{\alpha}^{2}} \sum_{\alpha} \epsilon_{\alpha}^{2} = 2. \]

The local equilibrium distribution function \( f_{\alpha}^{eq} \) is

\[ f_{\alpha}^{eq} = \begin{cases} 
\frac{h - hu_{\alpha}^{2}}{e^{2}} - \frac{gh^{2}}{2e^{2}}, & \alpha = 1, \\
\frac{gh^{2}}{4e^{2}} + \frac{hu_{\alpha}^{2}}{2e^{2}} + \frac{hu_{\alpha}}{2e}, & \alpha = 2, \\
\frac{gh^{2}}{4e^{2}} + \frac{hu_{\alpha}^{2}}{2e^{2}} - \frac{hu_{\alpha}}{2e}, & \alpha = 3.
\end{cases} \]

in terms of the distribution function, the macroscopic variables, the water depth \( h \) and the velocity \( u \) can be defined by

\[ h = \sum_{\alpha} f_{\alpha}, \quad u = \frac{1}{h} \sum_{\alpha} \epsilon_{\alpha} f_{\alpha}. \]

2.2 Treatment of the wet-dry interface

The lattice Boltzmann approach [19] is used to treat wet-dry interface. Firstly, the bed friction is straightforwardly included in this method. What’s more, this method, which builds the relationship between a dry cell and its adjacent wet cell by using the particle distribution functions, can well avoid \( h = 0 \) in formula (8) when a cell from dry turns to wet.
As shown in Fig. 1, \( d_1, d_2 \) are dry nodes, while \( w_1, w_2 \) are wet nodes at time \( t \). After one iteration or time step, \( f^{i+\Delta t}_0(d_2), f^{i+\Delta t}_1(d_2) \) and \( f^{i+\Delta t}_1(w_1) \) are unknown. If you want to know \( f^{i+\Delta t}_1(w_1) \), you have to calculate \( f^{i+\Delta t}_1(d_2) \). Although \( d_0 \) initially is dry, \( f^{i+\Delta t}_1(d_2) \) can be gained through non-equilibrium part of the particle distribution function \( f^{i+\Delta t}_1(d_2) \).

![Figure 1. Sketch of 1D wet-dry interface.](image)

Setting \( \Delta t = \varepsilon \) and applying the Taylor expansion to the first term on the left-hand side of (5) in time and space at point \((x, t)\) results in

\[
\varepsilon \left( \frac{\partial}{\partial t} + \varepsilon_0 \frac{\partial}{\partial x} \right) f_\alpha + \frac{\varepsilon^2}{2} \left( \frac{\partial}{\partial t} + \varepsilon_0 \frac{\partial}{\partial x} \right)^2 f_\alpha + o(\varepsilon^3) = \frac{1}{\tau}(f_\alpha - f^{(0)}_\alpha) + \frac{\Delta t}{\gamma e_2} \varepsilon_0 F,
\]

in which \( f^{(0)}_\alpha = f^{eq}_\alpha \), \( f_\alpha \) can be expanded around \( f^{(0)}_\alpha \) using the Chapman-Enskog expansion

\[
f_\alpha = f^{(0)}_\alpha + \varepsilon f^{(1)}_\alpha + o(\varepsilon^2).
\]

Ignoring the higher order of \( \varepsilon \), and substitution of \( (10) \) into \( (9) \) leads to

\[
\varepsilon \left( \frac{\partial}{\partial t} + \varepsilon_0 \frac{\partial}{\partial x} \right)(f^{(0)}_\alpha + \varepsilon f^{(1)}_\alpha) + \frac{\varepsilon^2}{2} \left( \frac{\partial}{\partial t} + \varepsilon_0 \frac{\partial}{\partial x} \right)^2 (f^{(0)}_\alpha + \varepsilon f^{(1)}_\alpha)
\]

\[
= \frac{1}{\tau} f^{(1)}_\alpha + \frac{\Delta t}{\gamma e_2} \varepsilon_0 F,
\]

and to order \( \varepsilon \) as

\[
\left( \frac{\partial}{\partial t} + \varepsilon_0 \frac{\partial}{\partial x} \right)f^{(0)}_\alpha = -\frac{1}{\tau} f^{(1)}_\alpha + \frac{\varepsilon_0 F}{2\varepsilon^2}.
\]

For the dry bed cell, considering

\[
f^{(0)}_\alpha = 0, \quad \frac{\partial f^{(0)}_\alpha}{\partial t} = 0.
\]

From \( (12), (13) \), it is easy to obtain

\[
f^{(1)}_\alpha = \frac{1}{\tau} \left( \frac{1}{\gamma e_2} \varepsilon_0 F - \varepsilon_0 \frac{\partial f^{(0)}_\alpha}{\partial x} \right).
\]

Substitution of \( (3), (13) \) and \( (14) \) into \( (10) \) results in

\[
f_\alpha = f^{(0)}_\alpha + \varepsilon f^{(1)}_\alpha = \varepsilon \tau \left[ \frac{1}{2\varepsilon^2} \varepsilon_0 \left( -gh \frac{\partial z_b}{\partial x} - \frac{\gamma e_2}{\rho} \right) - \varepsilon_0 \frac{\partial f^{(0)}_\alpha}{\partial x} \right]
\]

where

\[
\frac{\partial z_b}{\partial x} = \frac{z_b(x + \varepsilon_0 \Delta t) - z_b(x)}{\varepsilon_0 \Delta t},
\]

and

\[
\frac{\partial f^{(0)}_\alpha}{\partial x} = \frac{z_b(x)}{\varepsilon_0 \Delta t}.
\]
Replacing \( \tau \) with \( \Delta t \), substitution of (16), (17) into (15) gives

\[
\frac{\partial f^{(0)}_\alpha}{\partial x} = \frac{f^{(0)}_\alpha(x + e_\alpha \Delta t) - f^{(0)}_\alpha(x)}{e_\alpha \Delta t}.
\]

(17)

The equation (18) can be used to calculate \( f^\prime_0(d_2) \), provided that \( f^\prime_0(d_2) > 0 \), i.e. the fluid at \( w_1 \) has enough momentum to reach the neighboring dry node. Otherwise, the bounce-back scheme is used to compute \( f^\prime_0(d_2) \). \( f^\prime_0(d_2) \) can be determined by the average \( f^\prime_1(d_1) \) and \( f^\prime_1(w_1) \) as

\[
f^\prime_0(d_2) = \frac{f^\prime_1(d_1) + f^\prime_1(w_1)}{2}.
\]

(19)

3. Numerical simulation

A 1D tidal wave over a variable sloping bed is used in this paper. The example has been employed by other researchers [6, 19]. The length of the channel is 500 m, the slope (\( S = \partial z_0/\partial x \)) of the channel in the longitudinal is listed in Table 1. The initial water level of the channel is 1.75 m. The solid boundary is set at \( x = 0 \) m and the inlet boundary at \( x = 500 \) m is associated with the water depth (see Fig. 2) defined as

\[
h(500, t) = h_0 + \lambda \cos \left(2\pi \frac{t}{T}\right)
\]

(20)

in which \( h_0 = 1 \) m is the reference water surface, \( \lambda = 0.75 \) m is the amplitude of the tidal wave and \( T = 3600 \) s is the tidal period.

In the numerical computation, 200 lattices are used. \( \Delta t = 0.25 \) s, \( \Delta x = 9.5 \) m. Manning roughness coefficient is \( n = 0.03 \), the single relaxation time \( \tau = 0.7 \).

In order to test the effect of the bed friction on the tidal wave run-up, four values of \( C_b \) are given, i.e., \( C_b = 0 \), \( C_b = 0.01 \), \( C_b = 0.03 \), \( C_b = 0.05 \) and three time points, i.e., \( t = 0 \) min at the beginning, \( t = 12 \) min at wave run-down stage, \( t = 54 \) min at wave run-up stage are used to show different \( C_b \) values correspond to different water surface situations. The results are depicted in Fig. 3.
In Fig. 3(b), it shows that the maximum run-up height higher with the increase of the bed friction at the same moment in the descent process of waves, which states that the larger bed friction, the slower wave falling speed. In Fig. 3(c), it shows that the maximum run-up height becomes lower with the increase of the bed friction in the ascent process of waves, which also declares that the larger bed friction, the slower wave rising speed.

Figure 3. Comparisons of water surface for different values of $C_b$ at different time points: (a) $t = 0$ min; (b) $t = 12$ min; (c) $t = 54$ min.

4. Summary

In this article, we have presented a detailed numerical research of long wave run-up under the influence of bed friction using a lattice Boltzmann model for shallow water equations. The wetting and drying boundary is treated appropriately based on the theory of the lattice Boltzmann method by connecting the non-equilibrium particle distribution function of a dry cell with its adjoining wet cell. The bed friction, which is crucial to the wave run-up, has been directly incorporated into the lattice Boltzmann scheme. A tidal wave over three adverse slopes is used to explore the rule. The result indicates that the bigger bed friction, the stronger inhibiting effect on long wave run-up and run-down.

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6. References